

Stellar equilibrium in Einstein-Chern-Simons gravity

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Abstract

We consider a spherically symmetric internal solution within the context of Einstein-Chern-Simons gravity and derive a generalized five-dimensional Tolman-Oppenheimer-Volkoff (TOV) equation. It is shown that the generalized TOV equation leads, in a certain limit, to the standard five-dimensional TOV equation.

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I. INTRODUCTION

Some time ago was shown that the standard, five-dimensional General Relativity can be obtained from Chern-Simons gravity theory for a certain Lie algebra \mathfrak{B} [1], which was obtained from the AdS algebra and a particular semigroup S by means of the S-expansion procedure introduced in Refs. [2, 3].

The five-dimensional Chern-Simons Lagrangian for the \mathfrak{B} algebra is given by [1]

$$L_{\text{ChS}}^{(5)} = \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e + \alpha_3 \varepsilon_{abcde} \left(\frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right), \quad (1)$$

where l is a length scale in the theory (see [1]), $R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}$ is the curvature two-form with ω^{ab} the spin connection, and $T^a = D e^a$ with D the covariant derivative with respect to the Lorentz piece of the connection.

From (1) we can see that [1]:

- (i) the Lagrangian is split into two independent pieces, one proportional to α_1 and the other to α_3 . If one identifies the field e^a with the vielbein, the piece proportional to α_3 contains the Einstein-Hilbert term $\varepsilon_{abcde} R^{ab} e^c e^d e^e$ plus non-linear couplings between the curvature and the bosonic “matter” fields h^a and $k^{ab} = -k^{ba}$, which transform as a vector and as a tensor under local Lorentz transformations, respectively.
- (ii) In the strict limit where the coupling constant l equals zero we obtain solely the Einstein-Hilbert term in the Lagrangian [1].
- (iii) In the five-dimensional case, the connection of Eq. (17) of Ref. [1] has two possible candidates to be identified with the vielbein (see [4]), namely, the fields e^a and h^a , since both transform as vectors under local Lorentz transformations. Choosing e^a , makes the Einstein-Hilbert term to appear in the action, and $T^a = D e^a$ can be interpreted as the torsion two-form. This choice brings in the Einstein equations.

It is the purpose of this letter to find the stellar interior solution of the Einstein-Chern-Simons field equations, which were obtained in Refs. [5, 6].

We derive the generalized five-dimensional Tolman-Oppenheimer-Volkoff (TOV) equation and then we show that this generalized TOV equation leads, in a certain limit, to the standard five-dimensional TOV equation.

II. EINSTEIN-CHERN-SIMONS FIELD EQUATIONS FOR A SPHERICALLY SYMMETRIC METRIC

In Ref. [5] it was found that in the presence of matter described by the Lagrangian $L_M = L_M(e^a, h^a, \omega_{ab})$, we see that the corresponding field equations are given by

$$\begin{aligned}\varepsilon_{abcde} R^{cd} T^e &= 0, \\ \alpha_3 l^2 \varepsilon_{abcde} R^{bc} R^{de} &= -\frac{\delta L_M}{\delta h^a}, \\ \varepsilon_{abcde} (2\alpha_3 R^{bc} e^d e^e + \alpha_1 l^2 R^{bc} R^{de} + 2\alpha_3 l^2 D_\omega k^{bc} R^{de}) &= -\frac{\delta L_M}{\delta e^a}, \quad (2) \\ 2\varepsilon_{abcde} (\alpha_1 l^2 R^{cd} T^e + \alpha_3 l^2 D_\omega k^{cd} T^e + \alpha_3 e^c e^d T^e + \alpha_3 l^2 R^{cd} D_\omega h^e + \alpha_3 l^2 R^{cd} k_f^e e^f) &= -\frac{\delta L_M}{\delta \omega^{ab}}.\end{aligned}$$

For simplicity we will assume $T^a = 0$ and $k^{ab} = 0$. In this case the field equations (2) can be written in the form [6]

$$\begin{aligned}de^a + \omega_b^a e^b &= 0, \\ \varepsilon_{abcde} R^{cd} D_\omega h^e &= 0, \\ \alpha_3 l^2 \star (\varepsilon_{abcde} R^{bc} R^{de}) &= -\star \left(\frac{\delta L_M}{\delta h^a} \right), \quad (3) \\ \star (\varepsilon_{abcde} R^{bc} e^d e^e) + \frac{1}{2\alpha} l^2 \star (\varepsilon_{abcde} R^{bc} R^{de}) &= \kappa_E T_{ab} e^b,\end{aligned}$$

where $\alpha = \alpha_3/\alpha_a$, $\kappa_E = \kappa/2\alpha_3$, $T_{ab} = \star (\delta L_M/\delta e^a)$, “ \star ” is the Hodge star operator (see Appendix B) and T_{ab} is the energy-momentum tensor of matter fields (for details see Ref. [6]).

Since we are assuming spherical symmetry the metric will be of the form

$$ds^2 = -e^{2f(r)} dt^2 + e^{2g(r)} dr^2 + r^2 d\Omega_3^2 = \eta_{ab} e^a e^b \quad (4)$$

where $d\Omega_3^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2$ and $\eta_{ab} = \text{diag}(-1, +1, +1, +1, +1)$. The two unknown functions $f(r)$ and $g(r)$ will not turn out to be the same as in Ref. [6]. In Ref. [6] was found a spherically symmetric exterior solution, i.e. a solution where $\rho(r) = p(r) = 0$. Now $f(r)$ and $g(r)$ must satisfy the field equations inside the star, where $\rho(r) \neq 0$ and $p(r) \neq 0$. For this we need the energy-momentum tensor for the stellar material, which is taken to be a perfect fluid.

Introducing an orthonormal basis,

$$e^T = e^{f(r)} dt, \quad e^R = e^{g(r)} dr, \quad e^1 = r d\theta_1, \quad e^2 = r \sin \theta_1 d\theta_2, \quad e^3 = r \sin \theta_1 \sin \theta_2 d\theta_3.$$

Taking the exterior derivatives, using Cartan's first structural equation $T^a = de^a + \omega^a_b e^b = 0$ and the antisymmetry of the connection forms we find the non-zero connection forms. The use of Cartan's second structural equation permits to calculate the curvature matrix $R^a_b = d\omega^a_b + \omega^a_c \omega^c_b$.

Introducing these results in (3) and considering the energy-momentum tensor as the energy-momentum tensor of a perfect fluid at rest, i.e., $T_{TT} = \rho(r)$ and $T_{RR} = T_{ii} = p(r)$, where $\rho(r)$ and $p(r)$ are the energy density and pressure (for the perfect fluid), we find [6]

$$\frac{e^{-2g}}{r^2} (g'r + e^{2g} - 1) + \text{sgn}(\alpha) l^2 \frac{e^{-2g}}{r^3} g' (1 - e^{-2g}) = \frac{\kappa_E}{12} \rho, \quad (5)$$

$$\frac{e^{-2g}}{r^2} (f'r - e^{2g} + 1) + \text{sgn}(\alpha) l^2 \frac{e^{-2g}}{r^3} f' (1 - e^{-2g}) = \frac{\kappa_E}{12} p, \quad (6)$$

$$\begin{aligned} \frac{e^{-2g}}{r^2} \left\{ \left(-f'g'r^2 + f''r^2 + (f')^2 r^2 + 2f'r - 2g'r - e^{2g} + 1 \right) \right. \\ \left. + \text{sgn}(\alpha) l^2 \left(f'' + (f')^2 - f'g' - e^{-2g} f'' - e^{-2g} (f')^2 + 3e^{-2g} f'g' \right) \right\} = \frac{\kappa_E}{4} p. \end{aligned} \quad (7)$$

III. THE GENERALIZED TOLMAN-OPPENHEIMER-VOLKOFF EQUATION

Since when the torsion is null, the energy-momentum tensor satisfies the following condition (see Appendix B):

$$D_\omega(\star T_a) = 0, \quad (8)$$

we find that, for a spherically symmetric metric, (8) yields

$$f'(r) = -\frac{p'(r)}{\rho(r) + p(r)}, \quad (9)$$

an expression known as the hydrostatic equilibrium equation.

Following the usual procedure, we find that (5) has the following solution:

$$e^{-2g(r)} = 1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 l^2} \mathcal{M}(r)}, \quad (10)$$

where the Newtonian mass $\mathcal{M}(r)$ is given by

$$\mathcal{M}(r) = 2\pi^2 \int_0^r \rho(\bar{r}) \bar{r}^3 d\bar{r}. \quad (11)$$

On the other hand, from (6) we find that

$$\frac{df(r)}{dr} = f'(r) = \text{sgn}(\alpha) \frac{\kappa_E p(r) r^3 + 12r(1 - e^{-2g(r)})}{12l^2 e^{-2g(r)} (1 - e^{-2g(r)} + \text{sgn}(\alpha) \frac{r^2}{l^2})}. \quad (12)$$

Introducing (12) into (9) we find

$$\frac{dp(r)}{dr} = p'(r) = -\text{sgn}(\alpha) \frac{(\rho(r) + p(r)) (\kappa_E p(r)r^3 + 12r(1 - e^{-2g(r)}))}{12l^2 e^{-2g(r)} (1 - e^{-2g(r)} + \text{sgn}(\alpha) \frac{r^2}{l^2})} \quad (13)$$

and introducing (10) into (13) we obtain the generalized five-dimensional Tolman-Oppenheimer-Volkoff equation

$$\begin{aligned} \frac{dp(r)}{dr} = & -\frac{\kappa_E \mathcal{M}(r)\rho(r)}{12\pi^2 r^3} \left(1 + \frac{p(r)}{\rho(r)}\right) \left(1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} l^2 \mathcal{M}(r)\right)^{-1/2} \\ & \times \left[\frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)} - \frac{12 \text{sgn}(\alpha) \pi^2 r^4}{\kappa_E l^2 \mathcal{M}(r)} \left(1 - \sqrt{1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} l^2 \mathcal{M}(r)}\right) \right] \\ & \times \left[1 + \text{sgn}(\alpha) \frac{r^2}{l^2} \left(1 - \sqrt{1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} l^2 \mathcal{M}(r)}\right) \right]^{-1}. \end{aligned} \quad (14)$$

From (14) we can see that in the case of small l^2 limit, we can expand the root to first order in l^2 . In fact

$$\sqrt{1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} l^2 \mathcal{M}(r)} = 1 + \text{sgn}(\alpha) \frac{\kappa_E}{12\pi^2 r^4} l^2 \mathcal{M}(r) + \mathcal{O}(l^4). \quad (15)$$

Introducing (15) into (14) we find

$$\frac{dp(r)}{dr} \approx -\frac{\frac{\kappa_E \mathcal{M}(r)\rho(r)}{12\pi^2 r^3} \left(1 + \frac{p(r)}{\rho(r)}\right) \left(1 + \frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)}\right)}{\left(1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} l^2 \mathcal{M}(r)\right) \left(1 - \frac{\kappa_E}{12\pi^2 r^2} \mathcal{M}(r)\right)} \quad (16)$$

From (16) we can see that, in the limit where $l \rightarrow 0$, we obtain

$$\frac{dp(r)}{dr} = p'(r) \approx -\frac{\kappa_E \mathcal{M}(r)}{12\pi^2 r^3} \left(1 + \frac{p(r)}{\rho(r)}\right) \left(1 + \frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)}\right) \left(1 - \frac{\kappa}{12\pi^2 r^2} \mathcal{M}(r)\right)^{-1}, \quad (17)$$

which is the standard five-dimensional Tolman-Oppenheimer-Volkoff equation (see Eq. (A4)) (compare with the four-dimensional case shown in Ref. [7]).

To solve the generalized TOV equation (14), an equation of state relating ρ and p is needed. This equation should be supplemented by the boundary condition that $p(R) = 0$ where R is the radius of the star.

Given an equation of state $p(\rho)$, the problem can be formulated as a pair of first-order differential equations for $p(r)$, $\mathcal{M}(r)$ and $\rho(r)$, (14) and

$$\mathcal{M}'(r) = 2\pi^2 r^3 \rho(r), \quad (18)$$

with the initial condition $\mathcal{M}(0) = 0$. In addition, it is necessary to provide the initial condition $\rho(0) = \rho_0$.

Let us return to the problem of calculating the metric. Once we compute $\rho(r)$, $\mathcal{M}(r)$, and $p(r)$, we can immediately obtain $g(r)$ from (10) and $f(r)$ from (12)

$$f(r) = - \int_r^\infty \frac{\kappa_E \mathcal{M}(\bar{r})}{12\pi^2 \bar{r}^3} \left(1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 \bar{r}^4} l^2 \mathcal{M}(\bar{r}) \right)^{-1/2} \times \left[\frac{\pi^2 \bar{r}^4 p(\bar{r})}{\mathcal{M}(\bar{r})} - \frac{12 \text{sgn}(\alpha) \pi^2 \bar{r}^4}{\kappa_E l^2 \mathcal{M}(\bar{r})} \left(1 - \sqrt{1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 \bar{r}^4} l^2 \mathcal{M}(\bar{r})} \right) \right] \times \left[1 + \text{sgn}(\alpha) \frac{\bar{r}^2}{l^2} \left(1 - \sqrt{1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 \bar{r}^4} l^2 \mathcal{M}(\bar{r})} \right) \right]^{-1} d\bar{r} \quad (19)$$

where we have set $f(\infty) = 0$, a condition consistent with the asymptotic limit from the exterior solution.

It should be noted that if $r > R$, i.e., out of the star, the following conditions are satisfied:

$$\mathcal{M}(r) = M \quad , \quad p(r) = \rho(r) = 0. \quad (20)$$

Integrating (19) we find

$$f(r) = \frac{1}{2} \ln \left[1 + \text{sgn}(\alpha) \frac{r^2}{l^2} \left(1 - \sqrt{1 + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 r^4} l^2 M} \right) \right], \quad (21)$$

so that

$$e^{2f(r)} = e^{-2g(r)} = 1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 l^2} M}, \quad (22)$$

which coincides with the outer solution.

A. Constant Density: $\rho(r) = \rho_0$

We will now consider the solution of (14) in the case where the energy density is constant, $\rho(r) = \rho_0$, inside the star. In this case the hydrostatic equilibrium equation (9) can be directly integrated,

$$\rho_0 + p(r) = C e^{-f(r)}, \quad (23)$$

where C is an integration constant.

On the other hand, from (18) $\mathcal{M}(r)$ is given by

$$\mathcal{M}(r) = \frac{\pi^2}{2} \rho_0 r^4. \quad (24)$$

Introducing (24) into (10) we have

$$e^{-2g(r)} = 1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa_E}{12l^2} \rho_0 r^4}. \quad (25)$$

Now, let us add the field equations (5) and (6)

$$\frac{e^{-2g}}{r^3}(f' + g') [r^2 + \text{sgn}(\alpha)l^2 (1 - e^{-2g})] = \frac{\kappa_E}{12}(\rho_0 + p). \quad (26)$$

Using now (23), multiplying by e^{-g} and integrating we have

$$e^f = \frac{\kappa_E}{12} C e^{-g} \int \frac{r^3 dr}{e^{-3g} [r^2 + \text{sgn}(\alpha)l^2 (1 - e^{-2g})]} + C_0 e^{-g}, \quad (27)$$

where C_0 is the corresponding integration constant. Since

$$\int \frac{r^3 dr}{e^{-3g} [r^2 + \text{sgn}(\alpha)l^2 (1 - e^{-2g})]} = \frac{-\text{sgn}(\alpha)l^2 e^{g(r)}}{\sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0} (1 - \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0})} \quad (28)$$

we find

$$e^f = C_1 + C_0 e^{-g}, \quad (29)$$

where

$$C_1 := -\frac{\text{sgn}(\alpha)\kappa_E l^2 C}{12\sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0} (1 - \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0})}. \quad (30)$$

Then, we proceed to adjust the constants C , C_0 , and C_1 , so that the interior solution and exterior must match at $r = R$. In addition one should require that the pressure vanishes at $r = R$.

The calculations give

$$C = \rho_0 \sqrt{1 + \text{sgn}(\alpha)\frac{R^2}{l^2} \left(1 - \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}\right)}, \quad (31)$$

$$C_1 = -\frac{\text{sgn}(\alpha)\kappa_E l^2 \rho_0 \sqrt{1 + \text{sgn}(\alpha)\frac{R^2}{l^2} \left(1 - \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}\right)}}{12\sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0} (1 - \sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0})}, \quad (32)$$

and

$$C_0 = -\frac{1}{\sqrt{1 + \text{sgn}(\alpha)\frac{\kappa_E}{12}l^2\rho_0}}. \quad (33)$$

IV. SUMMARY AND OUTLOOK

We have considered a spherically symmetric internal solution within the context of Einstein-Chern-Simons gravity. We derived the generalized five-dimensional Tolman-Oppenheimer-Volkoff (TOV) equation and then we proved that this generalized TOV equation leads, in a certain limit, in the standard five-dimensional TOV equation.

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Appendix A: The standard Tolman-Oppenheimer-Volkoff equation in 5D

Let us recall that the energy-momentum tensor satisfies the condition

$$\nabla_\mu T^{\mu\nu} = 0. \quad (\text{A1})$$

If $T_{TT} = \rho(r)$ and $T_{RR} = T_{ii} = p(r)$ we find

$$\nabla_\mu T^{\mu r} = \frac{f'(r) \left(\rho(r) + p(r) \right) + p'(r)}{e^{2g(r)}},$$

so that

$$f' = -\frac{p'}{\rho + p}, \quad (\text{A2})$$

an expression known as the *hydrostatic equilibrium equation*.

From Eqs. (A10) and (A22) of Ref. [6] we find

$$f'(r) = \frac{\kappa_E \mathcal{M}(r)}{12\pi^2 r^3} \left(1 + \frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)} \right) \left(1 - \frac{\kappa_E}{12\pi^2 r^2} \mathcal{M}(r) \right)^{-1}. \quad (\text{A3})$$

Introducing (A2) into (A3) we obtain the standard five-dimensional *Tolman-Oppenheimer-Volkoff* equation

$$p'(r) = -\frac{\kappa_E \mathcal{M}(r)}{12\pi^2 r^3} \left(1 + \frac{p(r)}{\rho(r)} \right) \left(1 + \frac{\pi^2 r^4 p(r)}{\mathcal{M}(r)} \right) \left(1 - \frac{\kappa_E}{12\pi^2 r^2} \mathcal{M}(r) \right)^{-1}. \quad (\text{A4})$$

This may be compared with the four-dimensional case shown in equation (1.11.13) of reference [7].

Appendix B: Energy-momentum tensor

It is known that if the torsion is null, then the energy-momentum tensor is divergence-free, $\nabla_\mu T^{\mu\nu} = 0$. The 1-form energy-momentum is given by

$$\hat{T}_a := T_{\mu\nu} e_a^\mu dx^\nu. \quad (\text{B1})$$

Theorem. *If the energy-momentum tensor $T_{\mu\nu}$ and the 1-form energy-momentum \hat{T}_a are related by equation (B1), then in a torsion-free space-time*

$$\nabla_\mu T^\mu_\nu = -e_\nu^a \star D_\omega(\star \hat{T}_a) \quad (\text{B2})$$

Proof.

$$\star \hat{T}_a = \frac{\sqrt{-g}}{4!} \epsilon_{\mu\nu\rho\sigma\tau} T_a^\mu dx^\nu dx^\rho dx^\sigma dx^\tau. \quad (\text{B3})$$

After some algebra, we find

$$-e_\nu^a \star D_\omega(\star \hat{T}_a) = \frac{1}{\sqrt{-g}} \partial_\lambda (\sqrt{-g}) T_\nu^\lambda + \partial_\lambda T_\nu^\lambda - T_a^\lambda (\partial_\lambda e_\nu^a + \omega_{\lambda b}^a e_\nu^b), \quad (\text{B4})$$

and using the Weyl's lemma

$$\partial_\lambda e_\nu^a + \omega_{\lambda b}^a e_\nu^b - \Gamma_{\lambda\nu}^\rho e_\rho^a = 0, \quad (\text{B5})$$

we obtain

$$-e_\nu^a \star D_\omega(\star \hat{T}_a) = \frac{1}{\sqrt{-g}} \partial_\lambda (\sqrt{-g}) T_\nu^\rho + \partial_\lambda T_\nu^\lambda - \Gamma_{\lambda\nu}^\rho T_\rho^\lambda, \quad (\text{B6})$$

$$-e_\nu^a \star D_\omega(\star \hat{T}_a) = \partial_\lambda T_\nu^\lambda + \Gamma_{\lambda\rho}^\lambda T_\nu^\rho - \Gamma_{\lambda\nu}^\rho T_\rho^\lambda = \nabla_\lambda T_\nu^\lambda. \quad (\text{B7})$$

□

1. The Hodge star operator

The Hodge star operator for a p -form $P = \frac{1}{p!} P_{\alpha_1 \dots \alpha_p} dx^{\alpha_1} \dots dx^{\alpha_p}$ in a d -dimensional manifold with a non-singular metric tensor $g_{\mu\nu}$ is defined as

$$\star P = \frac{\sqrt{|g|}}{(d-p)!p!} \varepsilon_{\alpha_1 \dots \alpha_d} g^{\alpha_1 \beta_1} \dots g^{\alpha_p \beta_p} P_{\beta_1 \dots \beta_p} dx^{\alpha_{p+1}} \dots dx^{\alpha_d},$$

where $\varepsilon_{\alpha_1 \dots \alpha_d}$ is the total antisymmetric Levi-Civita tensor density of weight -1 .

2. Hydrostatic equilibrium equation

Let us consider a spherically and static-symmetric metric in five dimensions. The 1-form energy-momentum is given by

$$\hat{T}_a = T_{ab} e^b \quad (\text{B8})$$

where T_{ab} is the energy-momentum tensor in a comoving orthonormal frame. So, if the matter is a perfect fluid then

$$T_{TT} = \rho(r) \quad , \quad T_{RR} = T_{ii} = p(r). \quad (\text{B9})$$

Computing the conservation equation

$$D_\omega(\star \hat{T}_a) = 0 \quad (\text{B10})$$

we have

$$D_\omega(\star \hat{T}_a) = D_\omega(T_{ab} \star e^b) = \frac{1}{4!} \epsilon_{fbcd e} (D_\omega T_a^f) e^b e^c e^d e^e, \quad (\text{B11})$$

where we have used the torsion-free condition $D_\omega e^a = 0$. Therefore

$$D_\omega(\star \hat{T}_a) = \frac{1}{4!} \epsilon_{fbcd e} (dT_a^f + \omega_a^g T_g^f + \omega_g^f T_a^g) e^b e^c e^d e^e. \quad (\text{B12})$$

The calculations give

$$D_\omega(\star \hat{T}_R) = e^{-g} (p' + f'(\rho + p)) e^T e^R e^1 e^2 e^3 = 0 \quad (\text{B13})$$

from which we get the so-called hydrostatic equilibrium equation

$$p' + f'(\rho + p) = 0. \quad (\text{B14})$$

Appendix C: Dynamic of the field h^a

We consider now the field h^a . Expanding the field $h^a = h_\mu^a dx^\mu$ in their holonomic index we have [6]

$$h_a = h_{\mu\nu} e_a^\mu dx^\nu \quad (\text{C1})$$

For the space-time to be static and spherically symmetric, the field $h_{\mu\nu}$ must satisfy the Killing equation $\mathcal{L}_\xi h_{\mu\nu} = 0$ for $\xi_0 = \partial_t$ and the six generators of the sphere S_3 must be

$$\begin{aligned} \xi_0 &= \partial_t, \\ \xi_1 &= \partial_{\theta_3}, \\ \xi_2 &= \sin \theta_3 \partial_{\theta_2} + \cot \theta_2 \cos \theta_3 \partial_{\theta_3}, \\ \xi_3 &= \sin \theta_2 \sin \theta_3 \partial_{\theta_1} + \cot \theta_1 \cos \theta_2 \sin \theta_3 \partial_{\theta_2} + \cot \theta_1 \csc \theta_2 \cos \theta_3 \partial_{\theta_3} \\ \xi_4 &= \cos \theta_3 \partial_{\theta_2} - \cot \theta_2 \sin \theta_3 \partial_{\theta_3}, \\ \xi_5 &= \sin \theta_2 \cos \theta_3 \partial_{\theta_1} + \cot \theta_1 \cos \theta_2 \cos \theta_3 \partial_{\theta_2} - \cot \theta_1 \csc \theta_2 \sin \theta_3 \partial_{\theta_3}, \\ \xi_6 &= \cos \theta_2 \partial_{\theta_1} - \cot \theta_1 \sin \theta_2 \partial_{\theta_2}. \end{aligned} \quad (\text{C2})$$

Then, we have

$$\begin{aligned} h^T &= h_{tt}(r) e^T + h_{tr}(r) e^R, \\ h^R &= h_{rt}(r) e^T + h_{rr}(r) e^R, \\ h^i &= h(r) e^i. \end{aligned} \tag{C3}$$

From Eq. (3) we know that the dynamic of the field h^a is given by

$$\epsilon_{abcde} R^{cd} D h^e = 0 \tag{C4}$$

with

$$D h^a = d h^a + \omega^a_b h^b \tag{C5}$$

where

$$D h^T = e^{-g} (-h'_{tt} - f' h_{tt} + f' h_{rr}) e^T e^R, \tag{C6}$$

$$D h^R = e^{-g} (-h'_{rt} - f' h_{rt} + f' h_{tr}) e^T e^R, \tag{C7}$$

$$D h^i = \frac{e^{-g}}{r} (r h' + h - h_{rr}) e^R e^i - \frac{e^{-g}}{r} h_{rt} e^T e^i. \tag{C8}$$

Introducing (C6 - C8) into (C4) we have

$$h_{tr} = h_{rt} = 0, \tag{C9}$$

$$h_r = (r h)', \tag{C10}$$

$$h'_t = f'(h_r - h_t). \tag{C11}$$

To find solutions to (C9, C10, C11), we assume that $h_t(r)$ depends on r only through $f(r)$, namely

$$h_t(r) = h_t(f(r)) \tag{C12}$$

Introducing (C12) into (C11) we have

$$\frac{d h_t(f)}{d f} f'(r) = f'(h_r - h_t) \tag{C13}$$

from which we obtain the following linear differential equation, which is of first order and inhomogeneous:

$$\dot{h}_t + h_t = h_r, \tag{C14}$$

where $\dot{h}_t := \frac{dh_t(f)}{df}$. The homogeneous solution is given by

$$h_t^h(f) = Ae^{-f(r)}, \quad (\text{C15})$$

where A is a constant to be determined.

The particular solution depends on the shape of h_r . If we assume a functional relationship h with f , then the linearity of differential equation suggests the following ansatz:

$$h_r(r) = h_r(f(r)) = \sum_{n=0}^{\infty} B_n e^{nf(r)} + \sum_{m=2}^{\infty} C_m e^{-mf(r)}, \quad (\text{C16})$$

where B_n and C_m are real constants. So that the particular solution is given by

$$h_t^p(f) = \sum_{n=0}^{\infty} \frac{B_n}{n+1} e^{nf(r)} - \sum_{m=2}^{\infty} \frac{C_m}{m-1} e^{-mf(r)}. \quad (\text{C17})$$

Therefore the general solution is of the form

$$h_t(f(r)) = Ae^{-f(r)} + \sum_{n=0}^{\infty} \frac{B_n}{n+1} e^{nf(r)} - \sum_{m=2}^{\infty} \frac{C_m}{m-1} e^{-mf(r)}. \quad (\text{C18})$$

From (C10) we find

$$h(r) = \frac{1}{r} \left(\int h_r(r) dr + D \right), \quad (\text{C19})$$

where D is an integration constant. This means

$$h(r) = \frac{1}{r} \sum_{n=0}^{\infty} \left(B_n \int e^{nf(r)} dr \right) + \frac{1}{r} \sum_{m=2}^{\infty} \left(C_m \int e^{-mf(r)} dr \right) + \frac{D}{r}, \quad (\text{C20})$$

where A , B_n , and C_m are arbitrary constants, and $-e^{2f(r)}$ is the metric coefficient g_{00} .

1. Field asymptotically constant

Consider the simplest case where

$$h_r(r) = h = \text{constant} \quad (\text{C21})$$

in this case (C19) leads

$$h(r) = h + \frac{D}{r} \quad (\text{C22})$$

and

$$h_t(r) = Ae^{-f(r)} + h. \quad (\text{C23})$$

Since the vielbein is regular at $r = 0$ (center of the star), h^a should also be regularly at $r = 0$, i.e. we should have $D = 0$. Note that the coefficient $e^{f(r)}$ is regular at $r = 0$ as can be seen from (C19).

From (C20) we can see that the asymptotic behavior of the metric coefficients is given by

$$e^{2f(r \rightarrow \infty)} = e^{-2g(r \rightarrow \infty)} = 1. \quad (\text{C24})$$

Thus the asymptotic behavior of the field h^a is given by

$$h_r(r \rightarrow \infty) = h, \quad h(r \rightarrow \infty) = h, \quad h_t(r \rightarrow \infty) = A + h. \quad (\text{C25})$$

2. Constant density

If the density is constant then the inner solution is given by (25) and (29). In this case the solution for the field h^a is given by

$$h_r(r) = h, \quad h(r) = h \quad (\text{C26})$$

and

$$h_t(r) = \begin{cases} \frac{A}{C_0 + C_1 e^{-g(r)}} + h & \text{if } r < R, \\ \frac{A}{e^{-g(r)}} + h & \text{if } r \geq R, \end{cases} \quad (\text{C27})$$

where

$$e^{-g(r)} = \begin{cases} \sqrt{1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 l^2} \mathcal{M}(r)}} & \text{if } r < R \\ \sqrt{1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 l^2} M}} & \text{if } r \geq R \end{cases} \quad (\text{C28})$$

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